Improving the algorithm speed of solving the Lambert problem in identifying the initial required velocity of ballistic missiles

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Abstract. Problem Statement: The article deals with the Lambert problem for identifying the flight paths of an flying device between given points in a given time according to Kepler's law. This problem solving plays an important role in aircraft guidance and strategic missile guidance. Popular methods for solving the problem are presented in the book by Baetin [1]. However, one of the disadvantages of the process of solving Lambert problem is that there is a transcendental equation, which, when solved, uses a repeated method with a large number of operations or there is a difficulty in mathematical expression.
Research objective: The article proposes a new method to increase the algorithm speed of solving the Lambert problem. A ballistic missile is an object of study.
Research methods: The proposed method for solving the Lambert problem is based on the use of the secant method for finding the flying angle corresponding to the desired flying time.
Novelty: the new method allows to increase the speed of calculation, the number of repetitions decreases from 1000 times to 7 times in comparison with popular methods.
Results: The method is implemented in the MATLAB environment. The results of the study showed that the repetition process is simple and the convergence of the solution to the problem is faster, the solution is found only after several repetitions.
Keywords: Lambert problem, ballistic missile, strategic missile, transcendental equation, secant method.

1. Setting problem

The Lambert problem is to determine the flight trajectory between two given points over a specified time period, or the two-point boundary value problem. Solving this problem is not simple because it requires solving the transcendental equation over time. So far there have been many different methods proposed to solve this problem, but most of them use iterative methods such as the Stumpff function or superlobe function introduced by Battin [2].

In some cases,we do not wish to use iterative methods, especially in cases where the cost of a large calculation process is required. In addition to otonom situations on the cavity, the time for convergence is one of the very important factors. In order to solve the above problems, there are many different approaches proposed such as the linearization of the Lambert problem to update the nominal solution of the Lambert problem. For example, in [3], Arora and colleagues introduced the calculation process according to two different methods: Battin matrix transition method [4] and method of cosin transformation [5]. Similarly, Schumacher and colleagues [6] used a state-modifying matrix to deal with the uncertainty of the (orbit) problem of the Lambert problem.

One of the methods used to solve the Lambert problem is the scanning method. This method is very simple mathematically as well as when deployed, but the disadvantage of this method is to perform a lot of loops before converging on the desired solution. The article will propose a method to accelerate this algorithm by reducing the number of down-loop executions maximizingly, solving the Lambert problem.The results will be compared to prove the effectiveness of the proposed algorithm.

2. Method of implementation

a. Lambert problem and scanning algorithm

Considering an object under the force of gravity, it follows Newton's law as follows:

$$\ddot{x} = \frac{-gmx}{(x^2 + y^2)^{1,5}}, \tag{1}$$

$$\ddot{y} = \frac{-gmy}{(x^2 + y^2)^{1,5}}, \tag{2}$$

where x and y respectively are coordinates on X axis and Y axis in the Earth center coordinate system, and gm is attractive constant.

Assume that the original position of the object in the gravitational field that has coordinates:

$$x(0) = x_0, \tag{3}$$

$$y(0) = y_0, \tag{4}$$

and after t_F seconds, the object is at the position:

$$x(t_F) = x_F, \tag{5}$$

$$y(t_F) = y_F, \tag{6}$$

The Lambert problem will find the direction and magnitude (coordinates) of the initial velocity of the object in the gravitational field to satisfy the above boundary condition, that is, to determine the values of $\dot{x}(0)$ and $\dot{y}(0)$.

To solve the above problem, the article uses scanning algorithm. However, before applying this algorithm into Lambert problem. The article performs the analysis of the problem to implement the algorithm.

With flight angle and initial coordinates, the initial missile velocity required to meet the target on the Earth's surface is identified according to equation below [7, 8]:

$$V=\sqrt{\frac{gm(1-cos\phi)}{r_0\cos\gamma\left[\left(r_0\cos\dfrac{\gamma}{a}\right)-\cos(\phi+\gamma)\right]}}. \quad (7)$$

Where ϕ is center angle (angle between vector at the initial position of the missile and vector at the target position, γ is the initial angle of the missile's flight, a is the Earth's radius, and r_0 is initial distance from the Earth's center to the missile and is identified as follows:

$$r_0=a+alt. \quad (8)$$

With alt is initial altitude of the missile compared to the ground. The above equation is generalized for target at any r_F position, is rewritten as follows:

$$V=\sqrt{\frac{gm(1-cos\phi)}{r_0\cos\gamma\left[\left(r_0\cos\dfrac{\gamma}{r_F}\right)-\cos(\phi+\gamma)\right]}}, \quad (9)$$

where r_F is identified as follows:

$$r_0=a+alt_F \quad (10)$$

with alt_F is altitude of the target.

If the velocity vector is counterclockwise as shown in Figure 1 below, then the initial conditions for the velocity components in the coordinate system of the Earth's center is identified by the trigonometric equations as follows:

$$\dot{x}(0)=V\cos\left(\frac{\pi}{2}-\gamma+\theta_0\right), \quad (11)$$

$$\dot{y}(0)=V\sin\left(\frac{\pi}{2}-\gamma+\theta_0\right), \quad (12)$$

where γ is the angle representing the relative angle of the missile velocity compared to the tangent line at the intersection of the vector of the Earth's center to the initial position of the missile compared to the Earth's surface. We can see from Fig. 1, θ_0 angle is the initial angle position of the missile compared to the X axis of the Descartes coordinate system.

On the other hand, if the velocity vector is towards clockwise direction as in Fig. 2, the initial conditions of velocity components in the Earth's center coordinate system have the following forms:

$$\dot{x}(0) = V\cos\left(\gamma - \frac{\pi}{2} + \theta_0\right), \tag{13}$$

$$\dot{y}(0) = V\sin\left(\gamma - \frac{\pi}{2} + \theta_0\right). \tag{14}$$

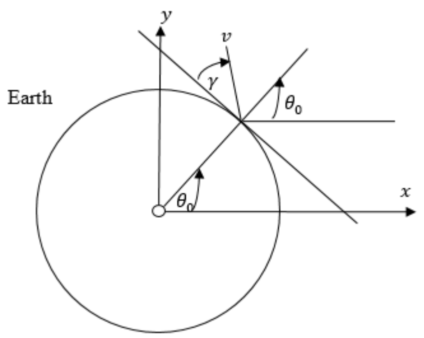

Figure 1. Flying counterclockwise

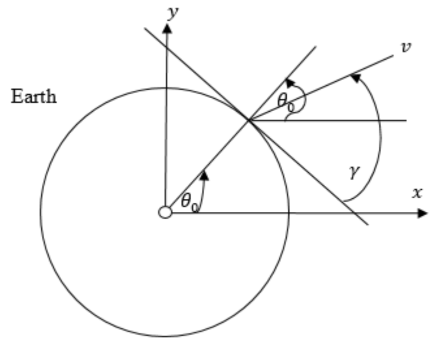

Figure 2. Flying clockwise

In document [8], an expression was given to identify the required time for the missile to meet target (t_F) as follows:

$$t_F = \left\{ \frac{r_0}{V\cos\gamma} \frac{\tan\gamma(1-\cos\phi)+(1-\lambda)\sin\phi}{(2-\lambda)\left[\frac{1-\cos\phi}{\lambda\cos^2\gamma}+\frac{\cos(\gamma+\phi)}{\cos\gamma}\right]} + \ldots + \frac{2\cos\gamma}{\lambda\left(\frac{2}{\lambda}-1\right)^{1.5}} \tan\left(\frac{\sqrt{\frac{2}{\lambda}-1}}{\cos\gamma\cot\frac{\phi}{2}-\sin\gamma}\right)\right\}, \tag{15}$$

where V is the initial required missile velocity to meet targets, λ is the constant identifying the conic section type [8]:

$$\lambda = \frac{r_0 V^2}{gm}. \tag{16}$$

First, we find the expression that identifies the center angle ϕ, consider Figure 3 presenting the initial position vector and the last position vector vector of the object in the gravitational field.

In Fig. 3, r_0 symbolizes for the vector from the Earth's center to the initial position of the object, r_F is the vector from the Earth's center to the last position of flying process of the object. The center angle is identified based on the definition of the scalar product of 2 vectors:

$$\phi = \cos^{-1} \frac{r_0 \cdot r_F}{|r_0| \cdot |r_F|}. \tag{17}$$

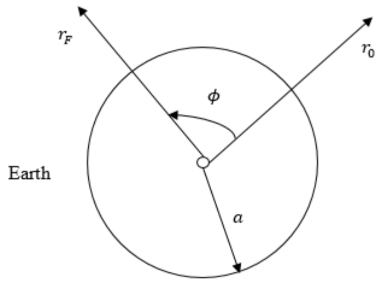

Figure 3. Center angle between the initial position vector and the last position vector

Therefore we had the sufficient information to solve the Lambert problem using scanning method.

If we know the initial position and the last position of the target, we can find the center angle ϕ according to (17). With a center angle ϕ, r_0, r_F and a flight path angle γ, we can identify the initial required velocity according analytic expression (9). Therefore we have γ, r_0 and r_F which are given in advanced with the following relations:

$$\phi = f(r_0, r_F), \tag{18}$$

$$V = f(r_0, r_F, \phi, \gamma), \tag{19}$$

$$t_F = f(V, \phi, \gamma). \tag{20}$$

As the Lambert problem stated in the previous part, we had r_0, r_F and t_F; needs to identify V and γ. If we use the above relations, we don't know how to choose γ or we don't guarantee that a particular angle γ will give the desired flying time. At that point, we use scanning method to solve the above problem. It means to find all the roots until we find the root satisfying all binding conditions of the problem. For example, starting with angle $\gamma = -90^{\circ}$, solve to find the velocity then solve to find the flying time. If the flying time is less than the desired flying time, we repeat that process with a greater value of. The search algorithm will stop when the calculated flying time is greater than the desired flying time. This method converged by [7] shows that monotonous time increases when flying angle increases.

The following is the simulation result for the above method with the simulation parameters as follows: desired flying time $t_{FDES} = 1000s$, the initial coordinate of the missile(in the Earth's center coordinate system) is $(4.5\times10^6;4.5\times10^6)$ m; target's coordinate (desired missile position at point t_{FDES}) in the Earth's center coordinate system is $(0;6.37\times10^6)$ m. The initial flying angle value to start the algorithm is $\gamma_0 = -90^o$.

From Fig. 4–6 we can see that the algorithm solving the Lambert problem converging towards the accurate root at the 1083rd iteration. However, we can achieve approximate root to asymptotic to the accurate root at the 334th iteration.

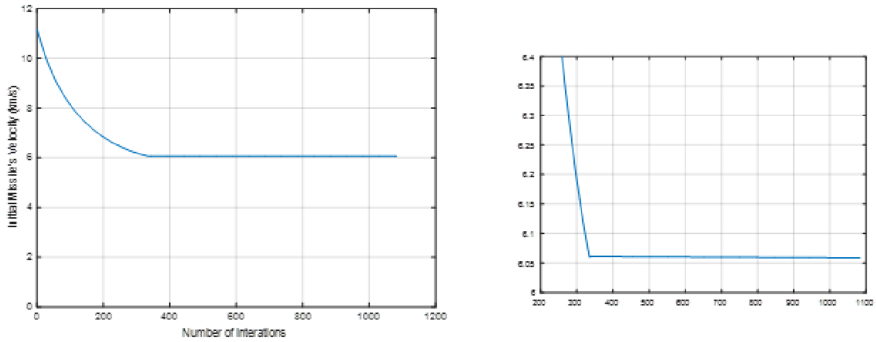

Figure 4. The initial required velocity of the missile

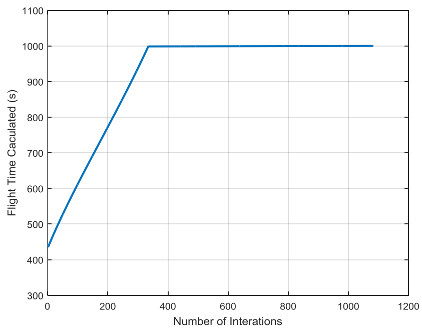

Figure 5. Flying time of the missile identified by the algorithm

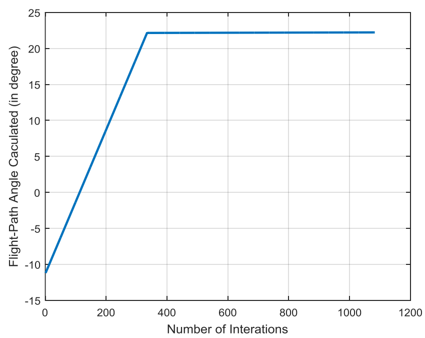

Figure 6. The initial flight angle of the missile

b. Proposing the method to accelerate the algorithm solving the Lambert problem

As we have shown through the simulation in the previous part, the algorithm solving the Lambert problem by the above scanning method have to implement different loops to find the appropriate flying path angle. In this part, the article will propose the method to reduce the loop maximumly to accelerate the algorithm. We consider the expression to identify the velocity in the previous part as follows:

$$V=\sqrt{\frac{gm(1-\cos\phi)}{r_0\cos\gamma\left[\left(r_0\cos\frac{\gamma}{r_F}\right)-\cos(\phi+\gamma)\right]}}. \tag{21}$$

Because the objects of flying device considered in the article are strategic missiles, therefore we have the value λ relating to releasing velocity [9]:

$$\lambda=2=\frac{V^2r_0}{gm}. \tag{22}$$

Substitute the above expression into (21) we achieve:

$$2=\frac{(1-\cos\phi)}{\cos\gamma\left[\left(r_0\cos\frac{\gamma}{r_F}\right)-\cos(\phi+\gamma)\right]}. \tag{23}$$

Solve the above equation to find the flying path angle γ. After algebraic transformations, we achieve the root corresponding to cases of minimum flight path angle and maximum flight path angle as follows [8]:

$$\gamma_{\min}=\tan^{-1}\frac{\sin\phi-\sqrt{\frac{2r_0}{r_F}(1-\cos\phi)}}{1-\cos\phi}, \tag{24}$$

$$\gamma_{\max}=\tan^{-1}\frac{\sin\phi+\sqrt{\frac{2r_0}{r_F}(1-\cos\phi)}}{1-\cos\phi}. \tag{25}$$

We can see that flight path angle parameter is limited in section $[\gamma_{\min},\gamma_{\max}]$. We won't implement the iteration of all values of flight path angle on this section, instead the article will use the secant search algorithm to find the flight path angle corresponding to the desired flying time according to the following expression:

$$\gamma_{n+1}=\gamma_n+\frac{(\gamma_n-\gamma_{n-1})(t_{FDES}-t_{F_n})}{t_{F_n}-t_{F_{n-1}}}. \tag{26}$$

From the above expression, we see that flying path angle γ_{n+1} is identified through the values at the previous point of time, which are γ_n and γ_{n-1}. The search algorithm will stop when t_{F_n} reaches sufficiently close to the desired flying time t_{FDES}.

To prove for the effectiveness of proposed method, the following is the simulation results with parameters of initial coordinate and target coordinate of the missile similarly to part 2 of the article. The flight path angle values will be calculated according (24) and (25).

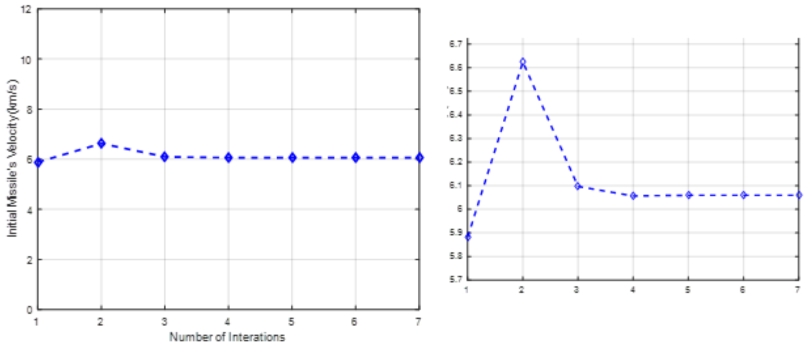

Figure 7. The initial required velocity of the missile

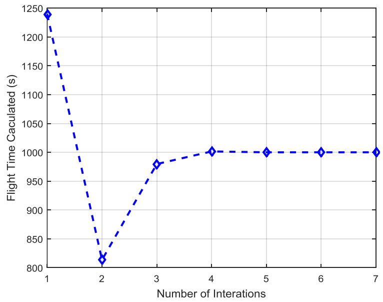

Figure 8. Flying time of the missile identified by the algorithm

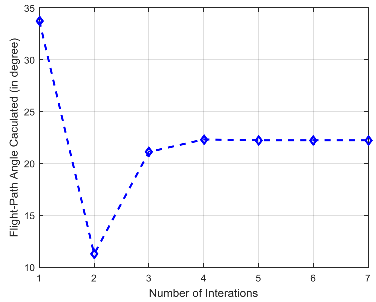

Figure 9. The initial flight angle of the missile

Comparing achieved results from the proposed iteration method (Figure 7–9) to the results from normal iteration method to solve Lambert problem, we can see that both method give the same result.

3. Result

The proposed iteration method gives the superiority more than solving the Lambert problem, it only takes 7 iterations to solve the Lambert problem while it takes 1000 iterations using the normal method.

4. Scientifical and practical meaning

The proposed algorithm suggests a new approach to solve the Lambert problem. This method is not drawn directly from the motion equations and does not need to use concepts related to the conic section or to solve the transcendental equations. However, this approach is totally natural, simple iteration process and the algorithm converges quickly in just a few loops.

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Повышение скорости решения задачи Ламберта при определении начальной требуемой скорости баллистических ракет

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Аннотация. Постановка задачи: В статье рассматривается задача Ламберта для определения траекторий полета летательного аппарата между заданными точками за заданное время по закону Кеплера. Решение задачи играет важную роль в наведении летательных аппаратов и наведении стратегической ракеты. Популярные методы решения задачи представлены в книге Баттин [1]. Однако, в системе уравнений существется трансцендентное уравнение, при решении которого используется повторный метод с большим количеством операций или возникает трудность в математическом выражении. Цель работы: В статье предложен новый метод для увеличения скорости решении задачи Ламберта. Баллистическая ракета является объект исследования. Методы исследова-

ния: Предложенный метод решения задачи Ламберта на основе использования алгоритма метода секущих для нахождения угла стрельбы к горизонту за заданное время. Новизна: новый метод позволяет увеличить скорость расчета, количество повторений уменьшается от 1000 раз до 7 раз по сравнению с популярными методами. Результаты: Метод реализован в среде MATLAB. Результаты исследования показали, что процесс повторений простой и сходимость решения задачи быстрее, решение найдено только через несколько раз повторений.
Ключевые слова: задача Ламберта, баллистическая ракета, стратегическая ракета, трансцендентное уравнение, метод секущих.

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